Phys 402
Fall 2022
Homework #0, Due Wednesday, 31 August, 2022

These are physics and math skills that you will need for Phys 402. You do not need to derive any of these results. However, you should be able to utilize all of these math skills in an exam situation. Hence, please review any concepts that present difficulty. Complete the following by hand (no assistance from computers!):

1. The imaginary unit is $i = \sqrt{-1}$, and satisfies $i^2 = -1$. Use the Euler formula to expand $e^{i\theta}$ for real θ.
2. Evaluate the value of $\frac{\sin x}{x}$ in the limit as $x \to 0$.
3. Given the three Cartesian unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$, calculate the following:
    a. $\hat{x} \times \hat{y}$
    b. $|\hat{x}|$
    c. $\hat{x} \cdot \hat{y}$
4. Given the vectors $\vec{r} = (r_x, r_y, r_z)$ and $\vec{s} = (s_x, s_y, s_z)$, calculate the cross product vector $\vec{r} \times \vec{s}$ in terms of its Cartesian components.
5. Find the eigenvalues and eigenvectors of this matrix: $\bar{\bar{A}} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. In other words find all the λ and $\vec{v}$ that satisfy $\bar{\bar{A}}\vec{v} = \lambda\vec{v}$ [For a review of linear algebra, see Appendix A of the Griffiths QM textbook.]
6. What is the determinant of $\bar{\bar{A}}$ and how is it related to the eigenvalues?
7. What is the trace of $\bar{\bar{A}}$ and how is it related to the eigenvalues?
8. Prove that the Pauli spin matrix $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ is *Hermitian*.
9. What does it mean if two vectors are "linearly independent"? What does it mean to have a set of vectors that "span a space"?
10. What is the general solution to the second-order linear differential equation $\ddot{x} = -\omega^2 x$, where ω is a real positive number?
11. What is the general solution to the second-order linear differential equation $\ddot{x} = +k^2 x$, where k is a real positive number?
12. Expand $y(x) = \ln(1 + x)$ to second order for $x \ll 1$.
13. Write the series expansion for $y(x) = \frac{1}{1-x}$ valid for $-1 < x < 1$.
14. Recall the Dirac delta function $\delta(x)$, which is defined through the expression $\int_{-\infty}^{\infty} f(x)\delta(x - a)dx = f(a)$ for any arbitrary (but well-behaved) function $f(x)$. Describe in words the properties of the delta function.

15. The Fourier transform of a function $f(x)$ can be written as $\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$. What is the Fourier transform of the Dirac delta function $\delta(x)$? Interpret the result.
16. Write out the expression for the overlap $\langle\chi|\psi\rangle$ between two quantum states in Hilbert space, $|\psi\rangle$ and $|\chi\rangle$, in the real-space representation.
17. Prove that the one-dimensional momentum operator $\hat{p} = -i\frac{d}{dx}$ is Hermitian. What can we say about the eigenvalues of any Hermitian operator?
18. Evaluate the commutator $[L^2, L_z]$ using the theorems proved in problem 4.22(b), and Eq. (4.99), of Griffiths and Schroeter Quantum Mechanics, 3rd Ed.
19. Write out the expression for the normalized spherical harmonic $Y_2^1(\theta,\varphi)$. Also, write down the mathematical statement of orthonormality of the spherical harmonics.
20. Using the figure below, express the spherical unit vectors $\hat{e}_r$, $\hat{e}_\theta$ and $\hat{e}_\phi$ in terms of the Cartesian unit vectors $\hat{x}$, $\hat{y}$, $\hat{z}$ for a general point in space labeled by (r,θ,ϕ).
21. Write down the differential volume element d^3r in spherical coordinates. Use the figure below for definition of the spherical coordinates.